

# Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem

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## Abstract

We give a new approach, inspired by Hörmander's  $L^2$ -method, to weighted variance inequalities which extend results obtained by Bobkov and Ledoux. It provides in particular a local proof of the dimensional functional forms of the Brunn-Minkowski inequalities. We also present several applications of these variance inequalities, including reverse Hölder inequalities for convex functions, weighted Brascamp-Lieb inequalities and sharp weighted Poincaré inequalities for generalized Cauchy measures.

## 1 Introduction

Our original motivation was to provide a local  $L^2$ -proof of the dimensional Prékopa inequality (Theorem 4 below). This inequality comes from a functional form of the Brunn-Minkowski inequality

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}, \quad (1.1)$$

where  $A$  and  $B$  are two Borel (later convex) subsets of  $\mathbb{R}^n$ , and  $A + B = \{a + b : a \in A \text{ and } b \in B\}$ . Eventually, we end up establishing the following two Theorems, which are the main new results of the present paper. In the sequel, we fix a Euclidean structure  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^n$  with respect to which Hessians  $D^2$  and gradients  $\nabla$  are computed; recall the notation  $\text{Var}_\mu(f) := \int f^2 d\mu - (\int f d\mu)^2$  for the variance of a function  $f$  with respect to a probability measure  $\mu$ .

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**Theorem 1.** Given  $\beta, r \in \mathbb{R}$  such that  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2 \in [n, +\infty)$ , set

$$A(n, \beta, r) := \frac{\beta - n}{n} - \frac{(n - 1)r^2}{n(\beta - 2r)} > 0.$$

Let  $\varphi$  be a positive  $C^2$  convex function defined on an open convex set  $\Omega \subseteq \mathbb{R}^n$ , such that  $d\mu_\beta = \varphi(x)^{-\beta} dx$  is a probability measure on  $\Omega$ . Then, for any locally Lipschitz  $f \in L^2(\mu_\beta)$ , setting  $g = f\varphi^{1-r}$ , we have

$$(\beta - 2r + 1) \text{Var}_{\mu_\beta}(f) \leq \int_{\Omega} \frac{\langle (D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_\beta + \frac{(1 - r)^2}{A(n, \beta, r)} \left( \int_{\Omega} f d\mu_\beta \right)^2. \quad (1.2)$$

In general, this theorem is applied with  $\Omega = \mathbb{R}^n$ . We have a parallel "concave" case (the reason will appear clearly later), which is:

**Theorem 2.** Given  $\beta, r \in \mathbb{R}$  satisfying  $\beta > -r + (-n + \sqrt{n^2 + 4(r^2 - r)n})/2 \in [-1, +\infty)$ , set

$$B(n, \beta, r) := \frac{\beta + n}{n} - \frac{(n - 1)r^2}{n(\beta + 2r)} > 0.$$

Let  $\varphi$  be a positive concave  $C^2$  function defined on a bounded open convex set  $\Omega \subset \mathbb{R}^n$  such that  $d\nu_\beta = \varphi(x)^\beta \mathbb{1}_\Omega(x) dx$  is a probability measure. Then, for any locally Lipschitz  $f \in L^2(\nu_\beta)$ , setting  $g = f\varphi^{1-r}$ , we have

$$(\beta + 2r - 1) \text{Var}_{\nu_\beta}(f) \leq \int_{\Omega} \frac{\langle (-D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\nu_\beta + \frac{(1 - r)^2}{B(n, \beta, r)} \left( \int_{\Omega} f d\nu_\beta \right)^2. \quad (1.3)$$

One can easily check that equality holds in (1.2) and (1.3) when  $f = \langle \nabla \varphi, z_0 \rangle \varphi^{r-1}$ , for some fixed  $z_0 \in \mathbb{R}^n$ .

In order to explain and motivate these results, but also to justify and understand the geometric nature of the conditions on the parameters, we need to step back a moment to the the Brunn-Minkowski inequalities.

Inequality (1.1) says that  $|\cdot|^{1/n}$  is concave; in the terminology recalled below, it means that the Lebesgue measure is  $1/n$ -concave. Using the homogeneity of Lebesgue measure, one can easily check that the inequality (1.1) is equivalent to the following *a-dimensional* inequality: for every  $A, B \in \mathbb{R}^n$  and  $t \in [0, 1]$ ,

$$|tA + (1 - t)B| \geq |A|^t |B|^{1-t}, \quad \forall t \in [0, 1]. \quad (1.4)$$

Inequality (1.4) says that Lebesgue measure  $|\cdot|$  on  $\mathbb{R}^n$  is log-concave. More generally, a Borel measure  $\mu$  on  $\mathbb{R}^n$  is said to be log-concave if it satisfies

$$\mu(tA + (1 - t)B) \geq \mu(A)^t \mu(B)^{1-t},$$

for any  $0 \leq t \leq 1$  and  $A, B$  two Borel sets of  $\mathbb{R}^n$ . The *dimensional* analogues of this property are defined as follows.

We introduce first, for  $\kappa \in \mathbb{R} \cup \{\pm\infty\}$ ,  $t \in [0, 1]$  and  $a, b \geq 0$ , the  $\kappa$ -mean

$$\mathcal{M}_t^\kappa(a, b) = (ta^\kappa + (1-t)b^\kappa)^{\frac{1}{\kappa}}$$

with the convention that  $\mathcal{M}_t^\kappa(a, b) = 0$  if  $ab = 0$ . The extremal cases are defined in the limit by  $\mathcal{M}_t^0(a, b) = a^t b^{1-t}$ ,  $\mathcal{M}_t^{-\infty}(a, b) = \min\{a, b\}$ , and  $\mathcal{M}_t^{+\infty}(a, b) = \max\{a, b\}$ . A Borel measure  $\mu$  on  $\mathbb{R}^n$  is said to be  $\kappa$ -concave, where  $-\infty \leq \kappa \leq +\infty$ , if it verifies the following inequality for all Borel sets  $A, B \subset \mathbb{R}^n$  and  $t \in [0, 1]$ :

$$\mu(tA + (1-t)B) \geq \mathcal{M}_t^\kappa(\mu(A), \mu(B)). \quad (1.5)$$

When  $\kappa = 0$ , then  $\mu$  is a log-concave measure, and the case  $\kappa = -\infty$  corresponds to the largest (by Hölder's inequality) class of measures, called *convex* or *hyperbolic* measures.

The characterization of  $\kappa$ -concave measures is given by the functional versions of the Brunn-Minkowski inequality. The functional form of the  $a$ -dimensional inequality (1.4) is the celebrated *Prékopa-Leindler inequality* [19, 21, 22], whereas the dimensional inequality (1.1) is associated to a family of inequalities, known as the *Borell-Brascamp-Lieb inequalities* (BBL in short) obtained in [11, 9]. Actually, many of the applications of the Brunn-Minkowski inequality and of their functional forms, the Prékopa-Leindler and BBL inequalities, can be obtained from the particular case where the sets  $A$  and  $B$  are convex, or when the functions under study are convex. So we shall state these particular cases only. The reader can find more background and applications in [17, 18].

The functional form of the  $a$ -dimensional inequality (1.4) for convex sets is the celebrated Prékopa inequality [22], which is the following particular case of the Prékopa-Leindler inequality.

**Theorem 3.** (Prékopa's inequality) *Let  $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex function. Then the function  $\phi$  defined on  $\mathbb{R}$  by*

$$e^{-\phi(t)} = \int_{\mathbb{R}^n} e^{-\varphi(t,x)} dx,$$

*is convex on  $\mathbb{R}$ .*

Note that we recover indeed the geometric result (1.4) when  $A$  and  $B$  are convex sets of  $\mathbb{R}^n$  by taking  $e^{-\varphi(t,x)} = \mathbf{1}_{(1-t)A+tB}(x)$ .

The corresponding dimensional version, relevant for the study of  $\kappa$ -concave measures with  $\kappa \neq 0$ , is the following particular case of the BBL inequality. Accordingly, we shall call it the dimensional Prékopa or Prékopa-BBL inequality. It contains two cases.

**Theorem 4.** (Prékopa-BBL or dimensional Prékopa inequality)

*First case: Let  $\varphi : \mathbb{R}^{n+1} \rightarrow (0, \infty]$  be a positive convex function and let  $\beta > n$ . Then the function  $\phi$  defined on  $\mathbb{R}$  by*

$$\phi(t) = \left( \int_{\mathbb{R}^n} \varphi(t, x)^{-\beta} dx \right)^{-\frac{1}{\beta-n}},$$

is convex.

*Second case:* let  $\varphi$  be a positive concave function on  $\Omega$ , a bounded open convex subset of  $\mathbb{R}^{n+1}$ , and let  $\beta \geq 0$ . Then the function  $\psi$  on  $\mathbb{R}$  defined by

$$\psi(t) = \left( \int_{\Omega(t)} \varphi(t, x)^\beta dx \right)^{\frac{1}{\beta+n}},$$

is concave, where  $\Omega(t) = \{x \in \mathbb{R}^n : (t, x) \in \Omega\}$ .

Of course, in the first statement of Theorem 4, by modifying  $\varphi$  if necessary, we can replace  $\mathbb{R}^{n+1}$  by any open convex subset  $\Omega$  and hence the integration is taken on the sections  $\Omega(t)$ .

Let us mention, for completeness, the geometric consequences of these inequalities in term of Brunn-Minkowski inequalities. Note that one needs the general BBL inequality if one wants (1.5) for all sets; with the particular case recalled above, the reader can check that one gets such inequality for convex sets  $A$  and  $B$ . It follows from the BBL inequality, and from a reverse statement of Borell (see [8, 9]), that a measure  $\mu$  on  $\mathbb{R}^n$  absolutely continuous with the Lebesgue measure is  $\kappa$ -concave (1.5) if and only if  $\kappa \leq \frac{1}{n}$  and  $\mu$  is supported on some (open) convex subset  $\Omega \subseteq \mathbb{R}^n$  where it has a positive density  $p(x)$  which satisfies, for all  $t \in (0, 1)$ ,

$$p(tx + (1-t)y) \geq \mathcal{M}_t^{\kappa_n}(p(x), p(y)), \quad \forall x, y \in \Omega, \quad (1.6)$$

where  $\kappa_n = \frac{\kappa}{1-n\kappa} \in [-\frac{1}{n}, +\infty]$  (equivalently,  $\kappa = \frac{\kappa_n}{1+n\kappa_n} \in [-\infty, \frac{1}{n}]$ ). In particular,  $\mu$  is log-concave if and only if it has a log-concave density ( $\kappa = \kappa_n = 0$ ), which is of course consistent with Prékopa's inequality. Note that the Lebesgue measure has the best possible concavity  $\kappa = \frac{1}{n}$  (which gives the Brunn-Minkowski inequality (1.1)) among convex measures, since a constant function satisfies (1.6) with  $\kappa_n = +\infty$ .

This description suggests two different behaviors, since, depending on the sign of  $\kappa_n$ ,  $p^{\kappa_n}$  is convex or concave (observe that  $\kappa_n$  is nonnegative if and only if  $\kappa \in [0, \frac{1}{n}]$ ). It was also noticed by Bobkov [2] that in the case  $\kappa \geq 0$ , the measures have bounded support. Since these two situations are present all along the paper (and already in the theorems above), let us clearly identify them:

**Case 1:** This corresponds to  $\kappa \leq 0$ . We set  $\beta = -\frac{1}{\kappa_n} = n - \frac{1}{\kappa} \geq n$  and we work with densities  $p(x) = \varphi(x)^{-\beta}$  where  $\varphi$  is a convex function on  $\mathbb{R}^n$  or on a subset  $\Omega$ . The typical examples are the (generalized) Cauchy probability measures given by

$$d\tau_\beta = \frac{1}{Z_\beta} (1 + |x|^2)^{-\beta} dx, \quad \beta > \frac{n}{2}, \quad (1.7)$$

where  $Z_\beta$  is a normalizing constant  $Z_\beta := \int_{\mathbb{R}^n} (1 + |x|^2)^{-\beta} dx = \pi^{\frac{n}{2}} \frac{\Gamma(\beta - \frac{n}{2})}{\Gamma(\beta)}$ .

**Case 2:** This corresponds to  $0 < \kappa \leq \frac{1}{n}$ . We set  $\beta = \frac{1}{\kappa_n} \in [0, +\infty)$ , although later we will also allow  $\beta \in (-1, +\infty)$ , and we work with densities  $p(x) = \varphi(x)^\beta$  where  $\varphi$  is a

concave function with compact support  $\Omega \subset \mathbb{R}^n$ . In this case, the typical examples are the probability measures given by

$$d\tau_{\sigma,\beta} = Z_{\sigma,\beta}^{-1}(\sigma^2 - |x|^2)^\beta \mathbb{1}_{\{|x| \leq \sigma\}} dx, \quad \beta \geq 0, \sigma > 0, \quad (1.8)$$

with normalizing constant  $Z_{\sigma,\beta}$  given by  $Z_{\sigma,\beta} = \sigma^{2\beta+n} \pi^{\frac{n}{2}} \frac{\Gamma(\beta+1)}{\Gamma(\beta+\frac{n}{2}+1)}$ .

Of major interests for us are the Poincaré-Sobolev inequalities that can be deduced from the functional forms of the Brunn-Minkowski inequalities above, as done by Bobkov and Ledoux in [3, 4, 5] by amplifying a linearization argument due to Maurey [20]. In [4] Bobkov and Ledoux explained how to derive from the Prékopa-Leindler inequality the so-called variance Brascamp-Lieb inequality [11] which states that for a log-concave probability measure  $d\mu = e^{-V}dx$ , with  $V$  smooth strictly convex on  $\mathbb{R}^n$ , one has, for every locally Lipschitz function  $f$ ,

$$\text{Var}_\mu(f) \leq \int_{\mathbb{R}^n} \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu. \quad (1.9)$$

The dimensional counterpart of this inequality, recently obtained in [3] as a consequence of the BBL inequality is as follows. Let  $\beta > n$  and let  $\mu_\beta$  be a probability measure on  $\mathbb{R}^n$  of the form  $d\mu_\beta = \varphi(x)^{-\beta} dx$  where  $\varphi$  is a positive convex function. Then, for any smooth function  $f$  on the support of  $\mu_\beta$ , setting  $g = \varphi f$ , we have

$$(\beta + 1) \text{Var}_{\mu_\beta}(f) \leq \int \frac{\langle (D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} d\mu_\beta + \frac{n}{\beta - n} \left( \int f d\mu_\beta \right)^2. \quad (1.10)$$

Note that this inequality corresponds to the particular case  $r = 0$  in (1.2) so that the connection between our main theorems and Brunn-Minkowski inequalities is now materializing.

Interestingly enough, the particular cases the Prékopa-Leindler and BBL inequalities recalled in Theorems 3 and 4 are sufficient to derive the two variance inequalities above. This is somehow at the heart of the local approaches to Prékopa's inequality (Theorem 3), amounting to compute  $\phi''(t)$ , as explained in [14]. Our original goal was to give such a local approach to the dimensional version (Theorem 4). As expected, the local variance inequality associated to Theorem 4 (for the first statement) is the inequality (1.10) obtained by Bobkov and Ledoux as a consequence of the BBL inequality.

So let us first explain in details the equivalence between the variance inequality (1.10) (together with (1.12) below) and the results of Theorem 4. Because there are two cases, corresponding to  $\kappa \leq 0$  or  $\kappa > 0$ , there will be two local variance inequalities. We will treat the case  $\kappa \leq 0$ ; the same arguments hold for the case  $\kappa > 0$ . By a direct computation, the second derivative of the function  $\phi(t)$  of Theorem 4 satisfies

$$\begin{aligned} \frac{\beta - n}{\beta} \frac{\phi''(t)}{\phi(t)} &= \int \frac{\partial_{tt}\varphi(t, x)}{\varphi(t, x)} d\mu_t(x) + \frac{n}{\beta - n} \left( \int \frac{\partial_t\varphi(t, x)}{\varphi(t, x)} d\mu_t(x) \right)^2 \\ &\quad - (\beta + 1) \text{Var}_{\mu_t} \left( \frac{\partial_t\varphi(t, \cdot)}{\varphi(t, \cdot)} \right), \end{aligned} \quad (1.11)$$

where  $\mu_t$  is the probability measure on  $\mathbb{R}^n$  given by

$$d\mu_t(x) = \frac{\varphi(t, x)^{-\beta} dx}{\int_{\mathbb{R}^n} \varphi(t, \cdot)^{-\beta} \cdot}.$$

In order to prove (1.10), we can assume for simplicity that  $\Omega$  is relatively compact and that  $f$  is smooth on  $\Omega$ . For  $g = \varphi f$ , and  $\epsilon > 0$ , the natural extension of  $\varphi$  with derivative  $g$  (to which we add a small uniformly convex factor for convenience) is the function

$$\varphi_\epsilon(t, x) := \varphi(x) + tg(x) + \frac{t^2}{2} \langle (D^2\varphi(x))^{-1} \nabla g(x), \nabla g(x) \rangle + \frac{\epsilon}{2} (|x|^2 + t^2).$$

This function is convex on  $\Omega \times (-a, a)$  for some  $a > 0$  small enough depending on  $\varphi, g, \epsilon$  and  $\Omega$ , since  $D^2\varphi(t, x)|_{t=0} \geq \epsilon \text{Id}$  on  $\Omega$ , and it satisfies

$$\varphi_{\epsilon}|_{t=0} = \varphi(x) + \epsilon|x|^2/2, \quad \partial_t \varphi_{\epsilon}|_{t=0} = g(x), \quad \partial_{tt}^2 \varphi_{\epsilon}|_{t=0} = \langle (D^2\varphi(x))^{-1} \nabla g(x), \nabla g(x) \rangle + \epsilon.$$

Theorem 4 tell us that the corresponding  $\phi = \phi_\epsilon$  is convex. Combining  $\phi''(0) \geq 0$  with (1.11) for  $\varphi_\epsilon(t, x)$ , and then letting  $\epsilon \rightarrow 0$ , we get, by uniform convergence on  $\Omega$ , the inequality (1.10). Conversely, in order to prove the first statement of Theorem 4, we can assume by approximation that  $\varphi$  is smooth and strictly convex in  $x$ . Then, if the inequality (1.10) holds, applying it with  $g := \partial_t \varphi|_{t=0}$  and using the fact that

$$\partial_{tt}^2 \varphi \geq \langle (D_x^2 \varphi)^{-1} \nabla_x \partial_t \varphi, \nabla_x \partial_t \varphi \rangle,$$

when  $\varphi$  is a convex function of  $(t, x)$  (strictly convex in  $x$ ), we get exactly that  $\phi''(t) \geq 0$ . We thus have shown that the inequality (1.10) is equivalent to the dimensional Prékopa's inequality in the case  $\kappa \leq 0$ .

Similarly, in the case  $\kappa > 0$ , the local form of the dimensional Prékopa inequality is the following variance inequality: Let  $\Omega$  be a bounded open convex subset of  $\mathbb{R}^n$  and let  $d\mu = \varphi(x)^\beta dx$  be a probability measure on  $\Omega$ , where  $\varphi$  is a positive concave function on  $\Omega$  and  $\beta \geq 0$ . Then for any smooth function  $f$  on  $\Omega$ , setting  $g = f\varphi$  one has:

$$(\beta - 1) \text{Var}_\mu(f) \leq \int_\Omega \frac{\langle (-D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} d\mu + \frac{n}{n + \beta} \left( \int_\Omega f d\mu \right)^2. \quad (1.12)$$

Therefore, it is sufficient to prove the inequalities (1.10) and (1.12) to prove the dimensional Prékopa's dimensional inequalities of Theorem 4. Our proof is based on Hörmander's  $L^2$ -method which is known to be useful in the context of variance inequalities and Prékopa's Theorem. Indeed, the Hörmander's  $L^2$ -method was first used for the local proof of Prékopa type inequalities by Cordero-Erausquin in [12], in connection with Berndtsson's complex generalization [1] of Prékopa's theorem. As we saw, and as explained in [14], the variance Brascamp-Lieb inequality (1.9) can clearly be identified as the local form of Prékopa's inequality, and this variance inequality can of course be proved by Hörmander's  $L^2$  method (it is exactly a real version of Hörmander's  $L^2$ -estimate)

The dimensional versions under study require, however, a few new arguments, as we shall see. And it allows for the more general statements given in Theorems 1 and 2. The inequalities (1.10) and (1.12) are particular cases of these theorems when we pick  $r = 0$ . The case  $r = 1$  is also of particular interest, as it amounts to weighted Brascamp-Lieb inequalities of the form

$$\mathrm{Var}_{\mu_\beta}(f) \leq \frac{1}{\beta - 1} \int_{\Omega} \langle (D^2\varphi)^{-1} \nabla f, \nabla f \rangle \varphi \, d\mu_\beta.$$

One can recover from this the classical Brascamp-Lieb inequality (1.9) for log-concave probability measures.

The rest of this paper is organized as follows. In the main, next section, we give the  $L^2$ -proof for the variance inequalities (1.2) and (1.3). In Section §3, we explain how these inequalities imply reverse Hölder inequalities for negative- $p$ -norms  $\|\varphi\|_{L^{-p}(dx)}$  of a convex function (Case 1), and  $p$ -norms in the case of a concave function (Case 2), as obtained by Borell in [10]; we also present a sharp bound for  $\mathrm{Var}_\mu(V)$  when  $d\mu = e^{-V(x)} dx$  is a log-concave measure on  $\mathbb{R}^n$ . Section §4 discusses weighted Brascamp-Lieb inequalities with application to log-concave measures. In Section §5, we derive sharp weighted Poincaré inequalities for generalized Cauchy type measures. In the last section §6, after some general comments, we explain how the results of the paper automatically extend to a Riemannian manifold  $M$  provided one introduces the correct Bakry-Emery type tensor associated to the Hessian of  $\varphi$  and the Ricci curvature of  $M$ .

## 2 The $L^2$ -proof of Theorems 1 and 2

In this section, we give the proof of the Theorems 1 and 2. It is inspired by Hörmander's  $L^2$ -duality method. Note that this gives, in particular, a new proof of the variance inequality (1.10) due to Bobkov and Ledoux, and of the inequality (1.12). We will detail the proof of Theorem 1. The proof of Theorem 2 is completely similar.

*Proof of Theorem 1.* Although the general argument is easy to follow, some of the formulas below are a bit long. The reader is encouraged to set  $r = 0$  in the present proof, which corresponds to the case of inequality (1.10). Formulas are nicer, and all the interesting ingredients are already at work in this particular case. Also, some of the formulas are complicated by the fact that we have a boundary term. Making formally  $\Omega = \mathbb{R}^n$  also simplifies things significantly.

In order to prove the inequality (1.2), we can assume, by standard approximation arguments, that the domain  $\Omega$  is bounded with  $C^\infty$ -smooth boundary, and  $\Omega$  is given by some  $C^\infty$ -smooth, convex function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$\Omega = \{x : \rho(x) < 0\}, \quad \text{and} \quad \nabla \rho \neq 0 \text{ on } \partial\Omega.$$

We shall denote

$$\nu(x) = \frac{\nabla \rho(x)}{|\nabla \rho(x)|}$$

the outer normal vector to  $\partial\Omega$  at the point  $x \in \partial\Omega$ . We can also assume that  $f$  and  $\varphi$  are  $C^\infty$  smooth on  $\overline{\Omega}$ .

Given  $\beta, r$  satisfying the condition of Theorem 1, it is easy to check that  $\beta > 2r$ . Let us introduce the operator  $L$  on  $L^2(\mu_\beta)$  given by

$$Lu = \varphi^r \Delta u - (\beta - r) \varphi^{r-1} \langle \nabla \varphi, \nabla u \rangle.$$

It is well defined for functions in  $C^2(\overline{\Omega})$  (we don't need to discuss the precise domain of  $L$  here). Integration by parts gives us that, for all  $u \in C^2(\overline{\Omega})$ , and  $v \in C^1(\overline{\Omega})$ ,

$$\int_{\Omega} v(x) Lu(x) d\mu_\beta = - \int_{\Omega} \langle \nabla u(x), \nabla v(x) \rangle \varphi(x)^r d\mu_\beta + \int_{\partial\Omega} \frac{\partial u(x)}{\partial \nu(x)} v(x) \varphi(x)^{-\beta+r} dx. \quad (2.1)$$

Next, we need to commute  $\nabla$  and  $L$ . It is readily checked that for any  $1 \leq i \leq n$  and  $u \in C^\infty(\overline{\Omega})$ ,

$$\partial_i Lu = L \partial_i u + r \varphi^{r-1} \partial_i \varphi \Delta u - (\beta - r) \varphi^{r-1} \sum_{j=1}^n \partial_{ij} \varphi \partial_j u - (\beta - r)(r - 1) \varphi^{r-2} \langle \nabla \varphi, \nabla u \rangle \partial_i \varphi.$$

Hence, if  $u$  is smooth on  $\overline{\Omega}$  and  $\frac{\partial u}{\partial \nu} = 0$  on  $\partial\Omega$ , one has

$$\int_{\Omega} (Lu)^2 d\mu_\beta = - \int_{\Omega} \langle \nabla Lu, \nabla u \rangle \varphi^r d\mu_\beta,$$

and therefore

$$\begin{aligned} \int_{\Omega} (Lu)^2 d\mu_\beta &= - \sum_{i=1}^n \int_{\Omega} L(\partial_i u) \varphi^r \partial_i u d\mu_\beta - r \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \Delta u \varphi^{2r} d\mu_\beta \\ &\quad + (\beta - r) \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta + (\beta - r)(r - 1) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta \\ &= r \int_{\Omega} \frac{\langle D^2 u \nabla u, \nabla \varphi \rangle}{\varphi} \varphi^{2r} d\mu_\beta - r \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \Delta u \varphi^{2r} d\mu_\beta \\ &\quad + (\beta - r) \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta + (\beta - r)(r - 1) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta \\ &\quad + \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta - \int_{\partial\Omega} \frac{\langle D^2 u \nabla u, \nabla \rho \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\Omega} \frac{\langle D^2 u \nabla u, \nabla \varphi \rangle}{\varphi} \varphi^{2r} d\mu_\beta &= \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \Delta u \varphi^{2r} d\mu_\beta + \frac{1}{\beta - 2r} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta \\ &\quad - \frac{1}{\beta - 2r} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta - \frac{1}{\beta - 2r} \int_{\partial\Omega} \frac{\langle D^2 u \nabla u, \nabla \rho \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx. \end{aligned}$$



Since  $\langle \nabla u(x), \nabla \rho(x) \rangle = 0$  on  $\partial\Omega$ , we have

$$\langle D^2 u(x) \nabla u(x), \nabla \rho(x) \rangle = -\langle D^2 \rho(x) \nabla u(x), \nabla u(x) \rangle, \quad \forall x \in \partial\Omega.$$

Therefore,

$$\begin{aligned} \int_{\Omega} (Lu)^2 d\mu &= \frac{\beta-r}{\beta-2r} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_{\beta} - \frac{r}{\beta-2r} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_{\beta} \\ &\quad + (\beta-r) \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_{\beta} + (\beta-r)(r-1) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_{\beta} \\ &\quad + \frac{\beta-r}{\beta-2r} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx. \end{aligned} \quad (2.2)$$

So fix now a smooth function  $f$  on  $\overline{\Omega}$  and denote  $\mu_{\beta}(f) := \int_{\Omega} f d\mu_{\beta}$ . We will use a classical fact concerning solution of the Laplace equation in  $L^2(\mu_{\beta-r})$  where

$$d\mu_{\beta-r} = \varphi^r d\mu_{\beta} = \varphi^{-(\beta-r)} dx = e^{-\log(\varphi^{\beta-r}(x))} dx$$

is a measure (not normalized) with smooth positive density on  $\overline{\Omega}$  (this will be the reason for which it is convenient to assume  $\Omega$  bounded and smooth). Namely, it follows from classical theory of elliptic equations, that given a smooth function  $F$  on  $\overline{\Omega}$  with  $\int F d\mu_{\beta-r} = 0$ , there exists a function  $u \in C^{\infty}(\overline{\Omega})$  with  $\frac{\partial u(x)}{\partial \nu(x)} = 0$  on  $\partial\Omega$  such that

$$Nu := \Delta u - \nabla[\log(\varphi^{\beta-r})] \cdot \nabla u = F.$$

We apply this result to  $F := (f - \mu_{\beta}(f)) \times \varphi^{-r}$ , and we get a function  $u \in C^{\infty}(\overline{\Omega})$  with  $\frac{\partial u(x)}{\partial \nu(x)} = 0$  on  $\partial\Omega$  such that

$$Lu = \varphi^r Nu = f - \mu_{\beta}(f).$$

We will use  $u$  to dualize the inequality.

Set  $\alpha = (\beta-1)/(\beta-2r+1)$ . We have

$$\text{Var}_{\mu_{\beta}}(f) = (1+\alpha) \int_{\Omega} (f - \mu_{\beta}(f)) Lu d\mu_{\beta} - \alpha \int_{\Omega} (Lu)^2 d\mu_{\beta}.$$

Since  $g = f\varphi^{1-r}$ , one has

$$\nabla f = \varphi^{r-1} \nabla g + (r-1) Lu \frac{\nabla \varphi}{\varphi} + (r-1) \mu_{\beta}(f) \frac{\nabla \varphi}{\varphi}.$$

Hence we have

$$\begin{aligned}
\text{Var}_{\mu_\beta}(f) &= -(1+\alpha) \int_{\Omega} \langle \nabla f, \nabla u \rangle \varphi^r d\mu_\beta - \alpha \int_{\Omega} (Lu)^2 d\mu_\beta \\
&= -(1+\alpha) \int_{\Omega} \frac{\langle \nabla u, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_\beta - (1+\alpha)(r-1) \int_{\Omega} Lu \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^r d\mu_\beta \\
&\quad - \alpha \frac{\beta-r}{\beta-2r} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta - (1+\alpha)(r-1) \mu_\beta(f) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^r d\mu_\beta \\
&\quad - \alpha(\beta-r)(r-1) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta - \alpha(\beta-r) \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta \\
&\quad + \alpha \frac{r}{\beta-2r} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta - \alpha \frac{\beta-r}{\beta-2r} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx \\
&= -(1+\alpha) \int_{\Omega} \frac{\langle \nabla u, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_\beta - \alpha(\beta-r) \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta \\
&\quad - \alpha \frac{\beta-r}{\beta-2r} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta - (1+\alpha)(r-1) \mu_\beta(f) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^r d\mu_\beta \\
&\quad - (1+\alpha)(r-1) \int_{\Omega} \Delta u \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta + (\beta-r)(r-1) \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta \\
&\quad + \alpha \frac{r}{\beta-2r} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta - \alpha \frac{\beta-r}{\beta-2r} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx. \tag{2.3}
\end{aligned}$$

We now calculate the term  $\int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta$ . Denote  $\gamma = \beta - 2r$ , it follows from the definition of  $L$  that

$$\begin{aligned}
\int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta &= \frac{1}{(\beta-r)^2} \left( \int_{\Omega} (Lu)^2 d\mu_\beta - \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta \right) \\
&\quad + \frac{2}{\beta-r} \int_{\Omega} \Delta u \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta \\
&= \frac{1}{\gamma(\gamma+r)} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta - \frac{1}{\gamma(\gamma+r)} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta \\
&\quad + \frac{1}{\gamma+r} \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta + \frac{2}{\gamma+r} \int_{\Omega} \Delta u \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta \\
&\quad + \frac{r-1}{\gamma+r} \int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta + \frac{1}{\gamma(\gamma+r)} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx.
\end{aligned}$$

Equivalently, we have

$$\begin{aligned}
\int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle^2}{\varphi^2} \varphi^{2r} d\mu_\beta &= \frac{1}{\gamma(\gamma+1)} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} d\mu_\beta - \frac{1}{\gamma(\gamma+1)} \int_{\Omega} (\Delta u)^2 \varphi^{2r} d\mu_\beta \\
&\quad + \frac{1}{\gamma+1} \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta + \frac{2}{\gamma+1} \int_{\Omega} \Delta u \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^{2r} d\mu_\beta \\
&\quad + \frac{1}{\gamma(\gamma+1)} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} dx. \tag{2.4}
\end{aligned}$$

It follows from (2.1) that  $\int_{\Omega} Lu \, d\mu_{\beta} = 0$ , or equivalently

$$\int_{\Omega} \frac{\langle \nabla \varphi, \nabla u \rangle}{\varphi} \varphi^r \, d\mu_{\beta} = \frac{1}{\beta - r} \int_{\Omega} \Delta u \, \varphi^r \, d\mu_{\beta}. \quad (2.5)$$

Combining (2.3), (2.4) and (2.5) with the value  $\alpha = (\beta - 1)/(\beta - 2r + 1)$ , one has

$$\begin{aligned} (\beta - 2r + 1) \operatorname{Var}_{\mu_{\beta}}(f) &= -2(\beta - r) \int_{\Omega} \frac{\langle \nabla u, \nabla g \rangle}{\varphi} \varphi^{2r} \, d\mu_{\beta} - (\beta - r)^2 \int_{\Omega} \frac{\langle D^2 \varphi \nabla u, \nabla u \rangle}{\varphi} \varphi^{2r} \, d\mu_{\beta} \\ &\quad - \frac{(\beta - r)^2}{\beta - 2r} \int_{\Omega} \|D^2 u\|_{HS}^2 \varphi^{2r} \, d\mu_{\beta} + \frac{\beta - 2r + r^2}{\beta - 2r} \int_{\Omega} (\Delta u)^2 \varphi^{2r} \, d\mu_{\beta} \\ &\quad - 2(r - 1) \mu_{\beta}(f) \int_{\Omega} \varphi^r \Delta u \, d\mu_{\beta} \\ &\quad - \frac{(\beta - r)^2}{\beta - 2r} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} \, dx. \end{aligned} \quad (2.6)$$

If  $\beta, r$  satisfy the condition of Theorem 1, then  $\beta > 2r$ , hence

$$\frac{(\beta - r)^2}{\beta - 2r} \int_{\partial\Omega} \frac{\langle D^2 \rho \nabla u, \nabla u \rangle}{|\nabla \rho|} \varphi^{-\beta+2r} \, dx \geq 0.$$

By using the pointwise estimates,  $2\langle v, w \rangle - \langle Hv, v \rangle \leq \langle H^{-1}w, w \rangle$  and  $(\operatorname{Tr}(Q))^2 \leq n\|Q\|_{HS}^2$  for two vector  $v, w \in \mathbb{R}^n$ ,  $H$  a positive  $n \times n$  matrix and  $Q$  a symmetric  $n \times n$  matrix, one gets

$$-2(\beta - r)\langle \nabla u, \nabla g \rangle - (\beta - r)^2 \langle D^2 \varphi \nabla u, \nabla u \rangle \leq \langle (D^2 \varphi)^{-1} \nabla g, \nabla g \rangle,$$

and, since  $\beta > 2r$ ,

$$-\frac{(\beta - r)^2}{\beta - 2r} \|D^2 u\|_{HS}^2 + \frac{\beta - 2r + r^2}{\beta - 2r} (\Delta u)^2 \leq -A(n, \beta, r) (\Delta u)^2.$$

Moreover, one has  $A(n, \beta, r) > 0$  when  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$  and so

$$\begin{aligned} (\beta - 2r + 1) \operatorname{Var}_{\mu_{\beta}}(f) &\leq \int_{\Omega} \frac{\langle (D^2 \varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} \, d\mu_{\beta} - A(n, \beta, r) \int_{\Omega} (\varphi^r \Delta u)^2 \, d\mu_{\beta} \\ &\quad - 2(r - 1) \mu_{\beta}(f) \int_{\omega} \varphi^r \Delta u \, d\mu_{\beta} \\ &\leq \int_{\Omega} \frac{\langle (D^2 \varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} \, d\mu_{\beta} + \frac{(1 - r)^2}{A(n, \beta, r)} \left( \int_{\Omega} f \, d\mu_{\beta} \right)^2. \end{aligned}$$

This finishes the proof of Theorem 1. □

### 3 Reverse Hölder inequalities and convexity

We give here some very elementary applications of Theorem 1 and Theorem 2 obtained by taking  $g = 1$  in the inequalities. We shall detail the applications of Theorem 1, and state some results without proof for Theorem 2.

So let us take  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$  and a convex function  $\varphi > 0$  on an open convex set  $\Omega \subseteq \mathbb{R}^n$  (we drop the normalization  $\int \varphi(x)^{-\beta} dx = 1$ ). Setting  $d\mu_\beta(x) = \frac{\varphi^{-\beta} dx}{\int \varphi^{-\beta}}$ , we can rewrite the inequality (1.2) as follows: when  $\varphi$  is smooth and  $f$  is a locally Lipschitz function  $f \in L^2(\mu_\beta)$ , we have

$$R(f) \leq \frac{1}{\beta - 2r + 1} \int_{\Omega} \frac{\langle (D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} d\mu_\beta, \quad (3.1)$$

with  $g = \varphi^{1-r} f$ , and

$$R(f) := \int_{\Omega} f^2 d\mu_\beta - \left(1 + \frac{(1-r)^2}{(\beta - 2r + 1)A(n, \beta, r)}\right) \left(\int_{\Omega} f d\mu_\beta\right)^2. \quad (3.2)$$

Observe that if we take the  $g$  identically one in (3.1), we get that  $R(\varphi^{r-1}) \leq 0$ . We deduce:

**Proposition 5.** *Let  $\Omega \subseteq \mathbb{R}^n$  be a convex open set and  $\varphi$  be a positive convex function on  $\Omega$ . If  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$ , then*

$$\int_{\Omega} \varphi^{-\beta} dx \int_{\Omega} \varphi^{-\beta-2(1-r)} dx \leq \left(1 + \frac{(1-r)^2}{(\beta - 2r + 1)A(n, \beta, r)}\right) \left(\int_{\Omega} \varphi^{-\beta-1+r} dx\right)^2. \quad (3.3)$$

In particular (case  $r = 0$ ), setting

$$\Psi(\beta) := \ln \left( \prod_{i=1}^n (\beta - i) \int_{\Omega} \varphi^{-\beta} dx \right) \quad (3.4)$$

we have

$$\Psi(\beta) + \Psi(\beta + 2) \leq 2\Psi(\beta + 1), \quad \forall \beta > n. \quad (3.5)$$

*Proof.* By approximation, we can assume that the convex function  $\varphi$  is smooth and strictly convex on  $\Omega$ . Then (3.3) is exactly the property  $R(\varphi^{r-1}) \leq 0$  that we deduced from plugging  $g = 1$  in (3.1). When  $r = 0$  the inequality rewrites as

$$\int_{\Omega} \varphi^{-\beta} dx \int_{\Omega} \varphi^{-\beta-2} dx \leq \frac{\beta(\beta - n + 1)}{(\beta + 1)(\beta - n)} \left(\int_{\Omega} \varphi^{-\beta-1} dx\right)^2, \quad (3.6)$$

which is equivalent to (3.5).  $\square$

It is interesting to note that there is equality in (3.5)-(3.6) when  $\varphi$  comes from a 1-homogeneous function, for instance in the following way. When  $\Omega = \mathbb{R}^n$ , if we take

$\varphi(x) = (1 + J_C(x))$  with  $J_C$  being the gauge of a convex body  $C \subset \mathbb{R}^n$ , then equality holds in (3.5)-(3.6). Indeed, one then has for  $\beta > n$ ,  $\int \varphi^{-\beta} = c_{n,\beta}|C|$  where  $c_{n,\beta}$  depends on  $n$  and  $\beta$  only, and therefore can be computed using  $\varphi(x) = 1 + |x|$ , for which one gets the relation (3.5) by elementary calculus. Note that this also shows that the constant  $A(n, \beta, 0) = \beta/(\beta - n)$  is optimal in (1.10). In the case  $\Omega$  is bounded, the argument works also when  $C$  is chosen to be a multiple of  $\overline{\Omega}$ .

Inequality (3.5) suggests that  $\Psi$  might be concave and is reminiscent of the Berwald type inequalities obtained by Borell (in the Case 2, see below). Let us point out that the concavity of  $\Psi$  is stated (without proof) by Bobkov and Madiman in [7]. A weaker, though useful, concavity can be deduced from (3.3) as follows. Let us define the function  $\psi$  on  $(n, \infty)$  by

$$\psi(\beta) = \ln \left( \int_{\Omega} \varphi^{-\beta} dx \right).$$

Inequality (3.3) is equivalent to

$$\psi(\beta) + \psi(\beta + 2(1 - r)) - 2\psi(\beta + 1 - r) \leq \ln \left( 1 + \frac{(1 - r)^2}{(\beta - 2r + 1)A(n, \beta, r)} \right), \quad (3.7)$$

for all  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$ . Since  $r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2 \rightarrow n + 1$  when  $r \rightarrow 1$ , we have for any  $\beta > n + 1$ , that inequality (3.7) holds for all  $r$  which is close enough to 1. Dividing the two sides of (3.7) by  $(1 - r)^2$  and then letting  $r \rightarrow 1$ , we get

$$\psi''(\beta) \leq \frac{n(\beta - 2)}{(\beta - 1)^2(\beta - n - 1)}, \quad \forall \beta > n + 1. \quad (3.8)$$

Therefore we have an upper bound for second derivative of the convex function  $\psi$  on  $(n + 1, \infty)$ . Moreover, it is readily checked that (3.8) is equivalent to the concavity of the function

$$\Phi(\beta) = \ln \left( (\beta - 1) \int_{\Omega} \varphi^{-\beta} dx \right) - \frac{n - 1}{n} \ln \left( \frac{(\beta - 1)^{\beta - 1}}{(\beta - n - 1)^{\beta - n - 1}} \right). \quad (3.9)$$

It is possible to improve inequality (3.8) in particular cases, when  $\varphi$  is smooth and strictly convex. Indeed, by using inequality (6.2) with functions of the form  $f = \varphi^{\alpha(r-1)}$ ,  $\alpha \neq 1$ , we can get the following inequality

$$\psi''(\beta) \leq \frac{W(\varphi, \beta)}{1 + W(\varphi, \beta)} \cdot \frac{n(\beta - 2)}{(\beta - 1)^2(\beta - n - 1)},$$

with  $W(\varphi, \beta) := \frac{(\beta-1)(\beta-n-1)}{n(\beta-2) \int_{\Omega} \varphi^{-\beta}} \int_{\Omega} \frac{\langle (D^2 \varphi)^{-1} \nabla \varphi, \nabla \varphi \rangle}{\varphi} \varphi^{-\beta} dx$ . This improves the bound (3.8) when  $W(\varphi, \beta) < \infty$ .

Inequality (3.8) is weaker, in general, than the concavity of  $\Psi$ , except in dimension  $n = 1$  where  $\Phi = \Psi$ . Nonetheless, it allows for a sharp variance estimate improving a result of Bobkov and Madiman.

**Corollary 6.** *Let  $d\mu = e^{-V(x)}dx$  be a log-concave probability measure on  $\mathbb{R}^n$ . Then*

$$\text{Var}_\mu(V) = \int V(x)^2 e^{-V(x)} dx - \left( \int V(x) e^{-V(x)} dx \right)^2 \leq n. \quad (3.10)$$

*Proof.* Note that with the notation  $\psi(\beta) = \log \int \varphi^{-\beta}$  we have  $\psi''(\beta) = \text{Var}_{\mu_\beta}(\ln \varphi)$ , where  $\mu_\beta$  is probability measure defined by

$$d\mu_\beta = \frac{\varphi(x)^{-\beta} dx}{\int e^{-\beta} dx}.$$

In our case, let  $V(x)$  be a convex function on  $\mathbb{R}^n$  such that  $\int e^{-V} dx = 1$ . Fix  $\beta_0 > n + 1$  and apply the inequality (3.8) to the convex function  $\varphi = e^{V/\beta_0}$  at  $\beta = \beta_0$ . We get

$$\int V(x)^2 e^{-V(x)} dx - \left( \int V(x) e^{-V(x)} dx \right)^2 \leq \beta_0^2 \frac{n(\beta_0 - 2)}{(\beta_0 - 1)^2(\beta_0 - n - 1)}.$$

Letting  $\beta_0$  tend to infinity, one gets the following variance inequality for  $V$

$$\int V(x)^2 e^{-V(x)} dx - \left( \int V(x) e^{-V(x)} dx \right)^2 \leq n.$$

as claimed. □

Inequality (3.10) was obtained in [6] by Bobkov and Madiman with an universal constant  $C \neq 1$  multiplying the  $n$ . Our version is sharp, as one can verify that there is equality in (3.10) for the exponential distribution  $e^{-\sum |x_i|}/2^n$ . Furthermore, note that when  $d\mu = e^{-V(x)}dx$  is an isotropic log-concave probability measure on  $\mathbb{R}^n$ , that is  $\int x e^{-V(x)} dx = 0$  and  $\int x \otimes x e^{-V(x)} dx = I_n$  then one has, by using the Cauchy-Schwartz inequality,

$$\begin{aligned} n &= \int \langle x, \nabla V(x) \rangle e^{-V(x)} dx \leq \left( \int |x|^2 e^{-V(x)} dx \right)^{\frac{1}{2}} \left( \int |\nabla V(x)|^2 e^{-V(x)} dx \right)^{\frac{1}{2}} \\ &= \sqrt{n} \left( \int |\nabla V(x)|^2 e^{-V(x)} dx \right)^{\frac{1}{2}}, \end{aligned}$$

and so the inequality (3.10) rewrites in this case as

$$\text{Var}_\mu(V) \leq \int |\nabla V(x)|^2 e^{-V(x)} dx.$$

We now mention similar consequences in the Case 2, that can be derived from Theorem 2. Recall that in this situation  $\varphi$  is a positive concave function on a *bounded*, open, convex subset  $\Omega \subset \mathbb{R}^n$  and introduce the probability measure supported on  $\Omega$

$$d\nu_\beta(x) = \frac{\varphi(x)^\beta \mathbb{1}_\Omega(x)}{\int_\Omega \varphi^\beta} dx$$

Note that  $\int_{\Omega} \varphi(x)^{\alpha} dx < \infty$ , for all  $\alpha > -1$ , and so  $\beta$  is a priori allowed to range in  $(-1, +\infty)$ . Let us denote

$$\overline{R}(f) = \int_{\Omega} f^2 d\nu_{\beta} - \left(1 + \frac{(1-r)^2}{(\beta+2r-1)B(n, \beta, r)}\right) \left(\int_{\Omega} f d\nu_{\beta}\right)^2, \quad (3.11)$$

As above, when  $\varphi$  is smooth, we rewrite (1.3) as  $(\beta+2r-1)\overline{R}(f) \leq \int_{\Omega} \frac{\langle (-D^2\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\nu_{\beta}$  for any smooth  $f$  and  $g = \varphi^{1-r} f$ . In particular, if we take  $f = \varphi^{r-1}$ ,  $g = 1$ , we get  $(\beta+2r-1)\overline{R}(\varphi^{r-1}) \leq 0$ . This of course extends to general  $\varphi$  and we get:

**Proposition 7.** *Let  $\Omega$  be a convex body of  $\mathbb{R}^n$  and  $\varphi$  be a positive convex function on  $\Omega$ . If  $\beta > \max\{-r + (-n + \sqrt{n^2 + 4(r^2 - r)n})/2, 1 - 2r\}$ , then*

$$\int_{\Omega} \varphi^{\beta} dx \int_{\Omega} \varphi^{\beta+2r-2} dx \leq \left(1 + \frac{(1-r)^2}{(\beta+2r-1)B(n, \beta, r)}\right) \left(\int_{\Omega} \varphi^{\beta+r-1} dx\right)^2. \quad (3.12)$$

In particular (case  $r = 0$ ), setting

$$\overline{\Psi}(\beta) := \ln \left( \prod_{i=1}^n (\beta + i) \int_{\Omega} \varphi(x)^{\beta} dx \right)$$

we have

$$\overline{\Psi}(\beta) + \overline{\Psi}(\beta + 2) \leq 2\overline{\Psi}(\beta + 1), \quad \forall \beta > -1. \quad (3.13)$$

Inequality (3.13) is a special case of a result of Borell [10] who proved that  $\overline{\Psi}$  is concave on  $(0, +\infty)$ . As before, one can show that inequality (3.12) implies, by letting  $r \rightarrow 1$ , the weaker result that the function  $\overline{\Phi}$  defined on  $(-1, \infty)$  by

$$\overline{\Phi}(\beta) = \ln \left( (\beta + 1) \int_{\Omega} \varphi^{\beta} dx \right) - \frac{n-1}{n} \ln \left( \frac{(\beta + 1)^{\beta+1}}{(\beta + n + 1)^{\beta+n+1}} \right)$$

is concave. Note however, that in dimension  $n = 1$  this reproduces and extends the result of Borell, since it gives the concavity of  $\Psi$  in the larger range  $(-1, +\infty)$ . Let us mention that the concavity of  $\overline{\Phi}$  in the form  $\overline{\Phi}'' \leq 0$  can be used to reproduce the inequality (3.10) as well.

## 4 Some weighted Brascamp-Lieb inequalities and applications

The following Brascamp-Lieb-type inequality can be derived from the Theorem 1,

**Theorem 8.** *Let  $\varphi$  be a  $C^2$ , positive, convex function defined on an (open) convex subset  $\Omega \subseteq \mathbb{R}^n$ . For any  $\beta > n$ , we denote  $\mu_{\beta}$  the probability measure on  $\Omega$  given by  $d\mu_{\beta}(x) = \frac{\varphi(x)^{-\beta}}{\int_{\Omega} \varphi^{-\beta} dx}$ . Then, when  $\beta \geq n+1$ , we have that for any locally Lipschitz function  $f \in L^2(\mu_{\beta})$ ,*

$$\text{Var}_{\mu_{\beta}}(f) \leq \frac{1}{\beta-1} \int_{\Omega} \langle (D^2\varphi)^{-1} \nabla f, \nabla f \rangle \varphi d\mu_{\beta}. \quad (4.1)$$

*Proof.* For the case  $\beta > n + 1$ , our result is proved by using the Theorem 1 with  $r = 1$  and function  $\tilde{\varphi} = c\varphi$  with  $c^\beta = \int_\Omega \varphi^{-\beta} dx$ .

The case  $\beta = n + 1$  is proved by letting  $\beta$  decrease to  $n + 1$ .  $\square$

Furthermore, one can derive from (4.1) applied to  $\beta + 1$ , after proper normalization, the following reverse-weighted inequality: for any locally Lipschitz function  $f$  on  $\Omega$ ,

$$\inf_{c \in \mathbb{R}} \int \frac{|f(x) - c|^2}{\varphi(x)} d\mu_\beta(x) \leq \frac{1}{\beta} \int \langle (D^2\varphi)^{-1} \nabla f, \nabla f \rangle d\mu_\beta, \quad \forall \beta \geq n. \quad (4.2)$$

Similarly, by applying the Theorem 2 to  $r = 1$ , one gets

**Theorem 9.** *Let  $\varphi$  be a positive concave function on a compact, convex set  $\Omega \subset \mathbb{R}^n$ . For  $\beta > -1$ , denote  $\nu_\beta$  the probability measure on  $\Omega$  defined by  $d\nu_\beta(x) = \frac{\varphi(x)^\beta}{\int_\Omega \varphi^\beta} dx$ . Then for any locally Lipschitz function  $f \in L^2(\nu_\beta)$ , we have*

$$\text{Var}_{\nu_\beta}(f) \leq \frac{1}{\beta + 1} \int_\Omega \langle (-D^2\varphi)^{-1} \nabla f, \nabla f \rangle \varphi d\nu_\beta. \quad (4.3)$$

Moreover, for any bounded, smooth function  $f$  on  $\Omega$  and  $\beta > 0$ , the following reversed-weighted form of (4.3) holds

$$\inf_{c \in \mathbb{R}} \int_\Omega \frac{|f(x) - c|^2}{\varphi} d\nu_\beta \leq \frac{1}{\beta} \int_\Omega \langle (-D^2\varphi)^{-1} \nabla f, \nabla f \rangle d\nu_\beta, \quad (4.4)$$

Inequalities (4.1) and (4.3) allow to simplify some arguments given by Bobkov and Ledoux [3] on how to recover the Brascamp-Lieb inequality (1.9).

Let  $d\mu = e^{-V(x)} dx$  be a log-concave probability measure on  $\mathbb{R}^n$ . We can assume that  $V$  is bounded from below. Hence  $1 + V/\beta$  is a positive, convex function for  $\beta$  large enough. Denoting

$$c_\beta = \left( \int (1 + \frac{V(x)}{\beta})^{-\beta} dx \right)^{\frac{1}{\beta}},$$

and applying inequality (4.1) to the convex function  $\varphi(x) = c_\beta(1 + V(x)/\beta)$  with  $\beta$  large enough, one obtains

$$\int f^2 (1 + \frac{V}{\beta})^{-\beta} \frac{dx}{c_\beta^\beta} - \left( \int f (1 + \frac{V}{\beta})^{-\beta} \frac{dx}{c_\beta^\beta} \right)^2 \leq \frac{\beta}{\beta - 1} \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle (1 + \frac{V}{\beta})^{-\beta+1} \frac{dx}{c_\beta^\beta},$$

for any bounded, smooth function  $f$  on  $\mathbb{R}^n$ . Letting  $\beta$  tend to infinity, one obtains (1.9) (since  $\lim_{\beta \rightarrow \infty} c_\beta^\beta = 1$ ).

We can use (4.3) as well. For this, consider the concave function  $1 - V/\beta$  on the domain  $\Omega_\beta = \{x \mid V(x) \leq \beta\}$ . If we let  $\beta \rightarrow \infty$  then we get again the inequality (1.9).

There is another way of obtaining (1.9) from (4.1) which is of independent interest and goes through an slight improvement of an inequality of Bobkov and Ledoux.



**Proposition 10.** *Let  $\mu$  be a log-concave probability measure on  $\mathbb{R}^n$  with density  $e^{-V}$  where  $V$  is a smooth convex function, and let  $\beta \geq n+1$ . For  $x \in \mathbb{R}^n$ , write  $W_x$  for the nonnegative self-adjoint operator*

$$W_x := D^2V(x) + \frac{1}{\beta} \nabla V(x) \otimes \nabla V(x),$$

*where  $a \otimes a$  denotes the linear map  $x \rightarrow \langle x, a \rangle a$ . Then, for any locally Lipschitz function  $f$  we have*

$$\text{Var}_\mu(f) \leq \frac{\beta}{\beta-1} \int \langle W^{-1} \nabla f, \nabla f \rangle d\mu. \quad (4.5)$$

*Proof.* For our  $\beta > n$ , we define a convex function  $\varphi = e^{V/\beta}$ . Then  $D^2\varphi = (\frac{1}{\beta} D^2V + \frac{1}{\beta^2} \nabla V \otimes \nabla V) e^{V/\beta}$ . Applying (4.1) for  $\beta \geq n+1$ , one gets the result.  $\square$

This result was first proved by Bobkov and Ledoux [3] with the (worst) constant  $C_\beta := (\sqrt{\beta+1} + 1)^2/\beta$  in place of  $\beta/(\beta-1)$  (although on the larger range  $\beta > n$ ). Since for the one-rank perturbation of a positive matrix we have

$$(A + a \otimes a)^{-1} = A^{-1} - \frac{A^{-1}a \otimes A^{-1}a}{1 + \langle A^{-1}a, a \rangle}, \quad (4.6)$$

we can rewrite the inequality as

$$\text{Var}_\mu(f) \leq \frac{\beta}{\beta-1} \left( \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle e^{-V} dx - \int \frac{\langle (D^2V)^{-1} \nabla V, \nabla f \rangle^2}{\beta + \langle (D^2V)^{-1} \nabla V, \nabla V \rangle} e^{-V} dx \right),$$

for any bounded, smooth function  $f$  on  $\mathbb{R}^n$ . And Brascamp-Lieb inequality (1.9) is deduced from (4.5) by letting  $\beta \rightarrow \infty$ .

As in [3], an application of (4.5) to Gaussian measures on  $\mathbb{R}^n$  gives us the weighted Poincaré type inequality for the family of  $\chi_n$ -distributions on  $[0, \infty)$  defined by

$$d\chi_n(r) = \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} r^{n-1} e^{-\frac{r^2}{2}} 1_{[0, \infty)}.$$

Indeed, for any bounded, smooth function  $g$  on  $[0, \infty)$ , setting  $f(x) = g(|x|)$  and  $\beta = n+1$ , inequality (4.5) with  $V(x) = (|x|^2 + n \ln 2\pi)/2$  yields

$$\text{Var}_{\chi_n}(g) \leq \frac{(n+1)^2}{n} \int_0^\infty \frac{(g'(r))^2}{n+1+r^2} d\chi_n \leq \int_0^\infty (g'(r))^2 \frac{n+3}{n+r^2} d\chi_n.$$

Another interesting application of inequality (4.5) concerns the probability measures on  $\mathbb{R}^n$  having the density  $d\mu_{r,n}(x) = c_{r,n} \exp\{-(|x_1|^r + \dots + |x_n|^r)/r\} dx$  with  $r \in [1, 2]$ . In particular, the result below reproduces the Poincaré inequality (although with non-optimal numerical constants) for such measures.

**Proposition 11.** *Let  $r \in [1, 2]$  and  $f$  be a smooth,  $\mu_{r,n}$ -square integrable function on  $\mathbb{R}^n$ . Then the following inequality holds:*

$$\begin{aligned} \int f(x)^2 d\mu_{r,n} - \left( \int f(x) d\mu_{r,n} \right)^2 &\leq 4 \int \sum_{i=1}^n \frac{|x_i|^{2-r}}{|x_i|^r + 2(r-1)} \left( \frac{\partial f}{\partial x_i}(x) \right)^2 d\mu_{r,n}(x) \\ &\leq C_r \int |\nabla f(x)|^2 d\mu_{r,n}, \end{aligned}$$

with  $C_r = \frac{4}{r}(2-r)^{\frac{2-r}{r}} \in (\frac{9}{5}, 4]$ .

*Proof.* Since  $\mu_{r,n} = \mu_{r,1} \otimes \cdots \otimes \mu_{r,1}$  it is enough, by elementary tenzoration, to prove the inequality in dimension  $n = 1$ . We consider the case  $r > 1$ ; the case  $r = 1$  follows by limit. Applying the inequality (4.5) for  $\beta = 2$  and to the convex function  $V(x) = |x|^r/r - \ln c_{r,1}$  on  $\mathbb{R}$ , one has

$$\int_{\mathbb{R}} f(x)^2 d\mu_{r,1} - \left( \int_{\mathbb{R}} f(x) d\mu_{r,1} \right)^2 \leq 4 \int_{\mathbb{R}} \frac{|x|^{2-r}}{2(r-1) + |x|^r} (f'(x))^2 d\mu_{r,1}.$$

Since  $1 < r \leq 2$  then the function  $g(t) = t^{2-r}/(2(r-1) + t^r)$  is bounded on  $[0, \infty)$ , with  $g(t) \leq (2-r)^{\frac{2-r}{r}}/r$  for all  $t \geq 0$ .  $\square$

## 5 Weighted Poincaré inequality for uniformly convex potentials with application to the Cauchy measures

This section discusses weighted Poincaré inequality for some special probability measures which are the Cauchy measures  $\tau_\beta$  defined by (1.7) and the measures  $\tau_{\sigma,\beta}$  defined by (1.8).

We observe first that if  $\varphi$  is uniformly convex on  $\Omega$ , that is  $D^2\varphi(x) \geq CI_n$  for all  $x \in \Omega$  and for some  $C > 0$ , where  $I_n$  denotes  $n \times n$  identity matrix, then we get from the Theorem 8 the following weighted Poincaré-type inequality and its reverse-weighted form.

**Theorem 12.** *Let  $\varphi$  be a positive, strictly convex function on an open convex set  $\Omega \subseteq \mathbb{R}^n$ , such that  $D^2\varphi \geq CI_n$ , for some  $C > 0$ . Introduce the probability measure on  $\Omega$  given by  $d\mu_\beta = \frac{\varphi(x)^{-\beta}}{\int_\Omega \varphi^{-\beta} dx}$ . Then, when  $\beta \geq n+1$ , we have, for any locally Lipschitz function  $f \in L^2(\mu_\beta)$ , that*

$$\text{Var}_{\mu_\beta}(f) \leq \frac{1}{C(\beta-1)} \int |\nabla f(x)|^2 \varphi(x) d\mu_\beta. \quad (5.1)$$

Moreover, for any  $\beta \geq n$  and for any smooth, bounded function  $f$  on  $\Omega$ , it holds

$$\inf_{c \in \mathbb{R}} \int \frac{|f(x) - c|^2}{\varphi(x)} d\mu_\beta(x) \leq \frac{1}{C\beta} \int |\nabla f|^2 d\mu_\beta. \quad (5.2)$$

It is well-known that the Poincaré inequality is equivalent to the exponential convergence of the semi-group with the generator associated to the Dirichlet form. Inequality (5.1) possesses a similar property, more precisely:

**Proposition 13.** *Let  $\varphi$  be a convex function satisfying the conditions of Theorem 12 and  $\beta \geq n + 1$ . Denote  $P_t = e^{tL_\beta}$  is the semigroup associated to the differential operator  $L_\beta := \varphi\Delta - (\beta - 1)\langle \nabla\varphi, \nabla\cdot \rangle$  on  $L^2(\mu_\beta)$ . Then inequality (5.1) is equivalent to*

$$\text{Var}_{\mu_\beta}(P_t f) \leq e^{-2C(\beta-1)t} \text{Var}_{\mu_\beta}(f), \quad (5.3)$$

for any  $f \in L^2(\mu_\beta)$ .

*Proof.* Since  $P_t 1 = 1$  and  $\int P_t f d\mu_\beta = \int f d\mu_\beta$ , then it is sufficient to prove this proposition for  $f \in L^2(\mu_\beta)$  and  $\int f d\mu_\beta = 0$ .

Assume that (5.1) holds. We define  $F(t) = \int (P_t f)^2 d\mu_\beta$ . Then the derivative of  $F$  satisfies  $F'(t) \leq -2C(\beta-1)F(t)$  by using the inequality (5.1). This inequality proves (5.3).

Conversely, assume that (5.3) holds. Since (5.3) becomes an equality at  $t = 0$ , differentiating the two sides of (5.3) at  $t = 0$  gives (5.1).  $\square$

Let us return to the Cauchy measures  $\tau_\beta$  defined by (1.7); for these measures, we have  $\varphi(x) = 1 + |x|^2$ , hence  $D^2\varphi = 2I_n$ . For instance, when  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$ , (1.2) takes the following form: for any locally Lipschitz  $f$  and  $g = f\varphi^{1-r}$ ,

$$(\beta - 2r + 1) \text{Var}_{\tau_\beta}(f) \leq \frac{1}{2} \int_{\Omega} |\nabla g|^2 \varphi^{2r-1} d\tau_\beta + \frac{(1-r)^2}{A(n, \beta, r)} \left( \int_{\Omega} f d\tau_\beta \right)^2.$$

Let us consider the particular case ( $r = 1$ ) given by the Theorem 12 and the Proposition 13:

**Corollary 14.** *Let  $\beta \geq n + 1$ . For any locally Lipschitz  $f \in L^2(\tau_\beta)$  we have*

$$\text{Var}_{\tau_\beta}(f) \leq \frac{1}{2(\beta - 1)} \int |\nabla f(x)|^2 (1 + |x|^2) d\mu_\beta. \quad (5.4)$$

Moreover, if  $\beta \geq n$  then

$$\inf_{c \in \mathbb{R}} \int \frac{|f(x) - c|^2}{1 + |x|^2} d\tau_\beta \leq \frac{1}{2\beta} \int |\nabla f(x)|^2 d\tau_\beta.$$

Finally, let us denote  $L_\beta = (1 + |x|^2)\Delta - 2(\beta - 1)\langle x, \nabla\cdot \rangle$  and  $P_t$  be the semigroup associated to  $L_\beta$  on  $L^2(\tau_\beta)$ , then

$$\text{Var}_{\tau_\beta}(P_t f) \leq e^{-4(\beta-1)t} \text{Var}_{\tau_\beta}(f)$$

for any  $f \in L^2(\tau_\beta)$  and  $\beta \geq n + 1$ .

The weighted Poincaré-type inequality (5.4) improves a result of Bobkov and Ledoux (Theorem 3.1, [3]). In that paper the authors obtained a similar result with the constant  $C_\beta = (\sqrt{1 + \frac{2}{\beta-1}} + \sqrt{\frac{2}{\beta+1}})^2$  in the place of 1 in the right hand side of (5.4). A simple calculation with the linear test functions  $f(x) = \langle v_0, x \rangle$  shows the constant  $1/2(\beta - 1)$  in (5.4) to be sharp. The disadvantage of the Theorem 14 compared with the result of Bobkov and Ledoux is that the domain of  $\beta$  is smaller, that is  $\beta \geq n + 1$  instead of  $\beta \geq n$  as in [3].

We now consider the case in which  $\varphi$  is positive, concave and  $\Omega$  is bounded. If  $-\varphi$  is strictly convex, it follows from the Proposition 9 and the same arguments in the Proposition 13 that

**Theorem 15.** Let  $\varphi$  be a positive, concave function on a bounded convex set  $\Omega \subset \mathbb{R}^n$  such that  $-D^2\varphi \geq CI_n$  for some  $C > 0$ . Introduce the probability measure on  $\Omega$  given by  $d\nu_\beta = \frac{\varphi^\beta(x)\mathbf{1}_\Omega(x)}{\int_\Omega \varphi^\beta} dx$ . Then, when  $\beta > -1$ , we have for any locally Lipschitz  $\nu_\beta$ -square integrable  $f$  on  $\Omega$ , that

$$\text{Var}_{\nu_\beta}(f) \leq \frac{1}{C(\beta+1)} \int_\Omega |\nabla f(x)|^2 \varphi(x) d\nu_\beta. \quad (5.5)$$

And for any bounded, smooth function  $f$  on  $\Omega$ ,

$$\inf_{c \in \mathbb{R}} \int_\Omega \frac{|f(x) - c|^2}{\varphi} d\nu_\beta \leq \frac{1}{C\beta} \int_\Omega |\nabla f(x)|^2 d\nu_\beta, \quad \forall \beta > 0. \quad (5.6)$$

Moreover, let us denote  $N_\beta = \varphi\Delta + (\beta+1)\langle \nabla\varphi, \nabla\cdot \rangle$  and let  $P_t$  be the semigroup associated to  $N_\beta$  on  $L^2(\nu_\beta)$ . Then we have

$$\text{Var}_{\nu_\beta}(P_t f) \leq e^{-2C(\beta+1)t} \text{Var}_{\nu_\beta}(f),$$

for any function  $f \in L^2(\nu_\beta)$  and  $\beta > -1$ .

Let us finally consider the measures  $\tau_{\sigma,\beta}$  defined by (1.8). For these measures, one has  $\varphi(x) = \sigma^2 - |x|^2$ . Since  $D^2\varphi = -2I_n$ , applying Theorem 15, we get the following results for measures  $\tau_{\sigma,\beta}$ ,

**Corollary 16.** Given  $\beta > -1$  and  $\sigma > 0$ , let  $\tau_{\sigma,\beta}$  be the probability measure defined by (1.8). For any locally Lipschitz,  $\tau_{\sigma,\beta}$ -square integrable functions  $f$  on  $\Omega = \{x : |x| < \sigma\}$ , we have

$$\text{Var}_{\tau_{\sigma,\beta}}(f) \leq \frac{1}{2(\beta+1)} \int_\Omega |\nabla f(x)|^2 (\sigma^2 - |x|^2) d\tau_{\sigma,\beta}. \quad (5.7)$$

Moreover, if  $\beta \geq 0$ , for any smooth function  $f$  on  $\Omega$ , we get

$$\inf_{c \in \mathbb{R}} \int_\Omega \frac{|f(x) - c|^2}{\sigma^2 - |x|^2} d\tau_{\sigma,\beta} \leq \frac{1}{2\beta} \int_\Omega |\nabla f(x)|^2 d\tau_{\sigma,\beta}.$$

Finally, consider  $N_{\sigma,\beta} = (\sigma^2 - |x|^2)\Delta - 2(\beta+1)\langle x, \nabla\cdot \rangle$  and let  $P_t = e^{tN_{\sigma,\beta}}$  be the semigroup associated to  $N_{\sigma,\beta}$  on  $L^2(\mu_{\sigma,\beta})$ ; then

$$\text{Var}_{\mu_{\sigma,\beta}}(P_t f) \leq e^{-4C(\beta+1)t} \text{Var}_{\mu_{\sigma,\beta}}(f),$$

for any function  $f \in L^2(\mu_{\sigma,\beta})$  and  $\beta > -1$ .

Let us remark that inequality (5.7) is sharp, and that equality holds for the linear functions  $f(x) = \langle v_0, x \rangle$ , for any  $v_0 \in \mathbb{R}^n$ .

## 6 Further remarks

We conclude with some straightforward extensions of Theorems 1 and 2.

One should note that the inequalities in these theorems are not invariant under translation. Consider first the Case 1, with  $\varphi$  convex smooth on some open convex set  $\Omega \subseteq \mathbb{R}^n$  and  $\beta > r + (n + \sqrt{n^2 + 4(r^2 - r)n})/2$ . Recall the definition of  $R(f)$  in (3.2). Next introduce

$$S(f) := \int_{\Omega} f \varphi^{r-1} d\mu_{\beta} - \left(1 + \frac{(1-r)^2}{(\beta - 2r + 1)A(n, \beta, r)}\right) \int_{\Omega} f d\mu_{\beta} \int_{\Omega} \varphi^{r-1} d\mu_{\beta},$$

so that

$$R(f + c\varphi^{r-1}) = R(f) + 2cS(f) + c^2R(\varphi^{r-1}), \quad \forall c \in \mathbb{R}.$$

Since the right hand side of (3.1) does not change if we replace  $f$  by  $f + c\varphi^{r-1}$ , we have

$$R(f) + 2cS(f) + c^2R(\varphi^{r-1}) \leq \frac{1}{\beta - 2r + 1} \int \frac{\langle (D^2\varphi)^{-1}\nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_{\beta}, \quad \forall c \in \mathbb{R} \quad (6.1)$$

with  $g = \varphi^{1-r}f$ . Optimizing the left hand side of (6.1) over  $c \in \mathbb{R}$ , we obtain a stronger version of the inequality (1.2),

$$R(f) - \frac{S(f)^2}{R(\varphi^{r-1})} \leq \frac{1}{\beta - 2r + 1} \int \frac{\langle (D^2\varphi)^{-1}\nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_{\beta}, \quad (6.2)$$

with  $g = \varphi^{1-r}f$ .

Similarly, in the Case 2, let  $\varphi$  be a concave function defined on a bounded open convex set  $\Omega \subset \mathbb{R}^n$  and  $\beta > -r + (-n + \sqrt{n^2 + 4(r^2 - r)n})/2$ . Recall the definition of  $\bar{R}(f)$  in (3.11) and introduce

$$\bar{S}(f) := \int_{\Omega} f \varphi^{r-1} d\nu_{\beta} - \left(1 + \frac{(1-r)^2}{(\beta + 2r - 1)B(n, \beta, r)}\right) \int_{\Omega} f d\nu_{\beta} \int_{\Omega} \varphi^{r-1} d\nu_{\beta}.$$

The optimized version of (1.3) is as follows (since in this case  $(\beta + 2r - 1)\bar{R}(\varphi^{r-1}) \leq 0$ ):

$$(\beta + 2r - 1) \left( \bar{R}(f) - \frac{\bar{S}(f)^2}{\bar{R}(\varphi^{r-1})} \right) \leq \int_{\Omega} \frac{\langle (-D^2\varphi)^{-1}\nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\nu_{\beta}, \quad (6.3)$$

with  $g = \varphi^{r-1}f$ .

A priori, these optimized forms leave room for different kind of normalization:  $\int f d\mu_{\beta} = 0$ , or  $\int f \varphi^{r-1} d\mu_{\beta} = 0$ . However, we were not able to obtain new information from it.

A nicer observation, maybe, is that the results in this paper automatically extend to Riemannian manifolds. This is one of the advantages of the  $L^2$  approach we exploited here.

Let  $M$  be an  $n$ -dimensional Riemannian manifold, equipped with its Riemannian element of volume  $d \text{ vol}$ . The only difference in the computations we did in Section §2 is an extra Ricci curvature term coming from the commutation of the (covariant) derivative and the Laplacian. More precisely, one can check, using the Bochner-Lichnerovitz formula (see for

instance Proposition 4.15 in [16]), that formula (2.2) holds with an extra term on the right equal to  $\frac{\beta-r}{\beta-2r} \int_M \text{Ric}(\nabla u, \nabla u) \varphi^{2r} d\mu_\beta$  where  $\text{Ric}_x(\cdot, \cdot)$  stands for the Ricci curvature at point  $x$  (that is also identified to a symmetric operator on the tangent space  $T_x M$ ). And then formula (2.6) holds with the term  $\frac{(\beta-r)^2}{\beta-2r} \int_M \text{Ric}(\nabla u, \nabla u) \varphi^{2r} d\mu_\beta$  added on its right hand side. Given a smooth function  $\varphi$  on  $M$ , introduce the symmetric operator, which can be seen as a modified Bakry-Emery tensor, defined on  $T_x M$  by

$$H_x \varphi := D^2 \varphi(x) + \frac{\varphi(x)}{\beta - 2r} \text{Ric}_x,$$

where  $D^2(x)\varphi$  denotes the Riemannian Hessian of  $\varphi$  at  $x$ . Then the result of this paper extend to  $M$  provided one properly replaces the convexity of  $\varphi$  (or of  $-\varphi$  for the Case 2). Here is an example of result, corresponding to Theorem 1

**Theorem 17.** *Let  $M$  be an  $n$ -dimensional manifold. Let us give constants  $\beta, r \in \mathbb{R}$  and  $A(n, \beta, r)$  as in Theorem 1. Assume that we are given a probability measure  $d\mu_\beta(x) = \varphi(x)^{-\beta} d\text{vol}(x)$  where  $\varphi$  a smooth function on  $M$  such that  $H_x \varphi > 0$  at every  $x \in M$ . Then for any locally Lipschitz  $\mu_\beta$ -square integrable function  $f$  on  $M$ , setting  $g = f\varphi^{1-r}$ , we have*

$$(\beta - 2r + 1) \text{Var}_{\mu_\beta}(f) \leq \int \frac{\langle (H\varphi)^{-1} \nabla g, \nabla g \rangle}{\varphi} \varphi^{2r} d\mu_\beta + \frac{(1-r)^2}{A(n, \beta, r)} \left( \int f d\mu_\beta \right)^2.$$

We leave to the reader the corresponding applications and particular cases given in Sections §3 and §4. Note that for the Case 2, for the measure  $d\nu_\beta = \varphi(x)^\beta d\text{vol}(x)$ , the operator to consider is

$$\tilde{H}_x \varphi := -D^2 \varphi(x) + \frac{\varphi(x)}{\beta + 2r} \text{Ric}_x,$$

and the concavity of  $\varphi$  is replaced by the requirement that  $\tilde{H}_x$  is positive for every  $x$  in a domain  $\Omega \subset M$ .

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